→ LU Decomposition

Theorem:

Let A be $n \times n$ matrix, we say that A have LU decomposition if there exist two matrices L and U such that:

1) L is a Lower triangular matrix $n \times n$ with diagonal entries (1)
2) U is a $n \times n$ upper triangular matrix
3) $A = LU$

Ex :

$$A = \begin{pmatrix} 2 & -1 & 3 \\ \boxed{4} & 2 & 1 \\ -6 & -1 & 2 \end{pmatrix}$$

Find the LU decomposition of A.

$R_2 - 2R_1 \to R_2$
$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ \boxed{-6} & -1 & 2 \end{pmatrix}$$

$$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Kelshi ta7t ldiagonal mn8ayerlo l sign

$R_3 + 3R_1 \to R_3$
$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & \boxed{-4} & 11 \end{pmatrix} \Rightarrow E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$R_3 + R_2 \to R_3$ $\underbrace{\begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{pmatrix}}_{U}$ $\Rightarrow$ $E_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$L = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix}}_{L}$ $A = LU$

Theorem:
If A is invertible then A has LU decomposition

Theorem:
(S) : $AX = b$
Solve (S) by using LU decomposition of A.
$A = LU$
$Ax = b \Rightarrow LUX = b$
Let $UX = y \Rightarrow Ly = b$

$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $\begin{cases} Ly = b \\ UX = y \end{cases}$

Ex: $\begin{cases} x_1 + 4x_2 + 3x_3 = 1 \\ x_1 + 3x_2 + 2x_3 = 2 \\ -x_1 + x_2 + x_3 = 3 \end{cases}$ solve the systeme using LU decomposition

$\begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ -1 & 1 & 1 \end{pmatrix}$ $R_2 - R_1 \to R_2$, $R_3 + R_1 \to R_3$ $\begin{pmatrix} 1 & 4 & 3 \\ 0 & -1 & -1 \\ 0 & 5 & 4 \end{pmatrix}$

$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$R_3 + 5R_2 \to R_3$ $\qquad \underbrace{\begin{pmatrix} 1 & 4 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{U} \Rightarrow E_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$

$$L = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -5 & 1 \end{pmatrix}}_{L}$$

$Ly = b$ $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$y_1 = 1$

$y_1 + y_2 = 2 \Rightarrow y_2 = 2 - 1 = 1$

$-y_1 - 5y_2 + y_3 = 3 \Rightarrow y_3 = +1 + 5 + 3 = 9$

$y = \begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix}$

$Ux = y$

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix}$$

$x_3 = -9$

$-x_2 + 9 = 1 \Rightarrow x_2 = 9 - 1 = 8$

$x_1 + 4(8) + 3(-9) = 1$

$\Rightarrow x_1 = -32 + 27 + 1 = -4$

$X = \begin{bmatrix} -4 \\ 8 \\ -9 \end{bmatrix}$